C*-ALGEBRAS GENERATED BY MULTIPLICATION OPERATORS AND COMPOSITION OPERATORS WITH RATIONAL SYMBOL

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ABSTRACT. Let R be a rational function of degree at least two, let J_R be the Julia set of R and let μ^L be the Lyubich measure of R. We study the C*-algebra $\mathcal{MC}_R$ generated by all multiplication operators by continuous functions in $C(J_R)$ and the composition operator C_R induced by R on $L^2(J_R, \mu^L)$. We show that the C*-algebra $\mathcal{MC}_R$ is isomorphic to the C*-algebra $\mathcal{O}_R(J_R)$ associated with the complex dynamical system $\{R^{\circ n}\}_{n=1}^{\infty}$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane and $H^2(\mathbb{D})$ the Hardy space of analytic functions whose power series have square-summable coefficients. For an analytic self-map φ on the unit disk $\mathbb{D}$, the composition operator C_φ on the Hardy space $H^2(\mathbb{D})$ is defined by $C_\varphi g = g \circ \varphi$ for $g \in H^2(\mathbb{D})$. Let $\mathbb{T}$ be the unit circle in the complex plane and $L^2(\mathbb{T})$ the square integrable measurable functions on $\mathbb{T}$ with respect to the normalized Lebesgue measure. The Hardy space $H^2(\mathbb{D})$ can be identified as the closed subspace of $L^2(\mathbb{T})$ consisting of the functions whose negative Fourier coefficients vanish. Let P_{H^2} be the projection from $L^2(\mathbb{T})$ onto the Hardy space $H^2(\mathbb{D})$. For $a \in L^\infty(\mathbb{T})$, the Toeplitz operator T_a on the Hardy space $H^2(\mathbb{D})$ is defined by $T_a f = P_{H^2} a f$ for $f \in H^2(\mathbb{D})$. Recently several authors considered C*-algebras generated by composition operators (and Toeplitz operators). Most of their studies have focused on composition operators induced by linear fractional maps ([6, 7, 13, 14, 15, 18, 20, 21, 22]).

There are some studies about C*-algebras generated by composition operators and Toeplitz operators for finite Blaschke products. Finite Blaschke products are examples of rational functions. For an analytic self-map φ on the unit disk $\mathbb{D}$, we denote by $\mathcal{TC}_\varphi$ the Toeplitz-composition C*-algebra generated by both the composition operator C_φ and the Toeplitz operator T_z. Its quotient algebra by the ideal $\mathcal{K}$ of the compact operators is denoted by $\mathcal{OC}_\varphi$. Let R be a finite Blaschke product of degree at least two with $R(0) = 0$. Watatani and the author [5] proved that the quotient algebra $\mathcal{OC}_R$ is isomorphic to the C*-algebra $\mathcal{O}_R(J_R)$ associated with the complex dynamical system introduced in [11]. In [4] we extend this result for general finite Blaschke products. Let R be a finite Blaschke product R of degree at least two. We showed that the quotient algebra $\mathcal{OC}_R$ is isomorphic to a certain Cuntz-Pimsner algebra and there is a relation between the quotient algebra $\mathcal{OC}_R$

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and the C*-algebra $\mathcal{O}_R(J_R)$. In general, two C*-algebras $\mathcal{OC}_R$ and $\mathcal{O}_R(J_R)$ are slightly different.

In this paper we give a relation between a C*-algebra containing a composition operator and the C*-algebra $\mathcal{O}_R(J_R)$ for a general rational function R of degree at least two. In the above studies we deal with composition operators on the Hardy space $H^2(\mathbb{D})$, while we consider composition operators on L^2 spaces in this case. Composition operators on L^2 spaces has been studied by many authors (see for example [23]). Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let φ a non-singular transformation on Ω. We define a measurable function by $C_\varphi f = f \circ \varphi$ for $f \in L^2(\Omega, \mathcal{F}, \mu)$. If C_φ is bounded operator on $L^2(\Omega, \mathcal{F}, \mu)$, we call C_φ the composition operator with φ.

Let R be a rational function of degree at least two. We consider the Julia set J_R of R, the Borel σ-algebra $\mathcal{B}(J_R)$ on J_R and the Lyubich measure μ^L of R. Let us denote by $\mathcal{MC}_R$ the C*-algebra generated by multiplication operators M_a for $a \in C(J_R)$ and the composition operator C_R on $L^2(J_R, \mathcal{B}(J_R), \mu^L)$. We regard the C*-algebra $\mathcal{MC}_R$ and multiplication operators as replacements of Toeplitz-composition C*-algbras and Toeplitz operators, respectively. We prove that the C*-algebra $\mathcal{MC}_R$ is isomorphic to the C*-algebra $\mathcal{O}_R(J_R)$ associated with the complex dynamical system.

There are two important points to prove this theorem. First one is to analyze operators of the form $C_R^* M_a C_R$ for $a \in C(J_R)$. We now consider a more general case. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and φ is a non-singular transformation. If C_φ is bounded, then we have $C_\varphi^* M_a C_\varphi = M_{\mathcal{L}_\varphi(a)}$ for $a \in L^\infty(\Omega, \mathcal{F}, \mu)$, where $\mathcal{L}_\varphi$ is the Frobenius-Perron operator for φ. This is an extension of covariant relations considered by Exel and Vershik [2]. Moreover similar relations have been studied on the Hardy space $H^2(\mathbb{D})$. Let φ be an inner function on $\mathbb{D}$. Jury showed a covariant relation $C_\varphi^* T_a C_\varphi = T_{A_\varphi(a)}$ for $a \in L^\infty(\mathbb{T})$, where A_φ is the Aleksandrov operator.

Second important point is an anaysis based on bases of Hilbert bimodules. In [4] and [5], a Toeplitz-composition C*-algebra for a finite Blaschke product R is isomorphic to a certain Cuntz-Pimsner algebra of a Hilbert bimodule X_R, using a finite basis of X_R. Let R be a rational function of degree at least two. The C*-algebra $\mathcal{O}_R(J_R)$ associated with complex dynamical system is defined as a Cuntz-Pimsner algebra of a Hilbert bimodule Y. Unlike the cases of [4] and [5], the Hilbert bimodule Y does not always have a *finite* basis. Kajiwara [9], however, constructed a concrete *countable* basis of Y. Thanks to this basis, we can prove the desired theorem.

2. Covariant relations

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\varphi : \Omega \to \Omega$ be a measurable transformation. Set $\varphi_*\mu(E) = \mu(\varphi^{-1}(E))$ for $E \in \mathcal{F}$. Then $\varphi_*\mu$ is a measure on Ω. The measurable transformation $\varphi : \Omega \to \Omega$ is said to be *non-singular* if $\varphi_*\mu(E) = 0$ whenever $\mu(E) = 0$ for $E \in \mathcal{F}$. If φ is non-singular, then $\varphi_*\mu$ is absolutely continuous with respect to μ. When μ is σ-finite, we denote by h_φ the Radon-Nikodym derivative $\frac{d\varphi_*\mu}{d\mu}$.

Let $1 \leq p \leq \infty$. We shall define the composition operator on $L^p(\Omega, \mathcal{F}, \mu)$. Every non-singular transformation $\varphi : \Omega \to \Omega$ induces a linear operator C_φ from $L^F p(\Omega, \mathcal{F}, \mu)$ to the linear space of all measurable functions on $(\Omega, \mathcal{F}, \mu)$ defined as $C_\varphi f = f \circ \varphi$ for $f \in L^p(\Omega, \mathcal{F}, \mu)$. If $C_\varphi : L^p(\Omega, \mathcal{F}, \mu) \to L^p(\Omega, \mathcal{F}, \mu)$ is bounded,

it is called a *composition operator* on $L^p(\Omega, \mathcal{F}, \mu)$ induced by φ. Let $(\Omega, \mathcal{F}, \mu)$ be σ-finite. For $1 \le p < \infty$, C_φ is bounded on $L^p(\Omega, \mathcal{F}, \mu)$ if and only if the Radon-Nikodym derivative h_φ is bounded (see for example [23, Theorem 2.1.1]). If C_φ is bounded on $L^p(\Omega, \mathcal{F}, \mu)$ for some $1 \le p < \infty$, then C_φ is bounded on $L^p(\Omega, \mathcal{F}, \mu)$ for any $1 \le p < \infty$ since h_φ is independent of p. For $p = \infty$, C_φ is bounded on $L^\infty(\Omega, \mathcal{F}, \mu)$ for any non-singular transformation.

Definition. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space, let $\varphi : \Omega \to \Omega$ be a non-singular transformation and let $f \in L^1(\Omega, \mathcal{F}, \mu)$. We define $\nu_{\varphi,f}$ by

$$\nu_{\varphi,f}(E) = \int_{\varphi^{-1}(E)} f d\mu$$

for $E \in \mathcal{F}$. Then $\nu_{\varphi,f}$ is an absolutely continuous measure with respect to μ. By the Radon-Nikodym theorem, there exists $\mathcal{L}_\varphi(f) \in L^1(\Omega, \mathcal{F}, \mu)$ such that

$$\int_E \mathcal{L}_\varphi(f) d\mu = \int_{\varphi^{-1}(E)} f d\mu$$

for $E \in \mathcal{F}$. We can regard $\mathcal{L}_\varphi$ as a bounded operator on $L^1(\Omega, \mathcal{F}, \mu)$ (see for example [16, Proposition 3.1.1]). We call $\mathcal{L}_\varphi$ the *Frobenius-Perron operator.*

Lemma 2.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let $\varphi : \Omega \to \Omega$ be a non-singular transformation. Suppose that $C_\varphi : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ is bounded. Then the restriction $\mathcal{L}_\varphi|_{L^\infty(\Omega,\mathcal{F},\mu)}$ is a bounded operator on $L^\infty(\Omega, \mathcal{F}, \mu)$ and $C_\varphi^* = \mathcal{L}_\varphi|_{L^\infty(\Omega,\mathcal{F},\mu)}$.*

Proof. Let $f \in L^\infty(\Omega, \mathcal{F}, \mu)$. First we shall show $\mathcal{L}_\varphi(f) \in L^\infty(\Omega, \mathcal{F}, \mu)$. There exists $M > 0$ such that $|f| \le M$. It follows from [16, Proposition 3.1.1] that $|\mathcal{L}_\varphi(f)| \le \mathcal{L}_\varphi(|f|) \le M\mathcal{L}_\varphi(1)$. Since $\mathcal{L}_\varphi(1) = h_\varphi$ and C_φ is bounded on $L^1(\Omega, \mathcal{F}, \mu)$, we have $\mathcal{L}_\varphi(1) \in L^\infty(\Omega, \mathcal{F}, \mu)$. Hence $\mathcal{L}_\varphi(f) \in L^\infty(\Omega, \mathcal{F}, \mu)$.

By the definition of $\mathcal{L}_\varphi$, we have

$$\int_\Omega \chi_E \mathcal{L}_\varphi(f) d\mu = \int_\Omega \chi_{\varphi^{-1}(E)} f d\mu = \int_\Omega (C_\varphi \chi_E) f d\mu$$

for $E \in \mathcal{F}$, where χ_E and $\chi_{\varphi^{-1}(E)}$ are characteristic functions on E and $\varphi^{-1}(E)$ respectively. Since C_φ is bounded on $L^1(\Omega, \mathcal{F}, \mu)$ and the set of integrable simple functions is dense in $L^1(\Omega, \mathcal{F}, \mu)$, the restriction map $\mathcal{L}_\varphi|_{L^\infty(\Omega,\mathcal{F},\mu)}$ is bounded on $L^\infty(\Omega, \mathcal{F}, \mu)$ and $C_\varphi^* = \mathcal{L}_\varphi|_{L^\infty(\Omega,\mathcal{F},\mu)}$. □

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $\varphi : \Omega \to \Omega$ a non-singular transformation. We consider the restriction of $\mathcal{L}_\varphi$ to $L^\infty(\Omega, \mathcal{F}, \mu)$. From now on, we use the same notation $\mathcal{L}_\varphi$ if no confusion can arise.

For $a \in L^\infty(\Omega, \mathcal{F}, \mu)$, we define the multiplication operator M_a on $L^2(\Omega, \mathcal{F}, \mu)$ by $M_a f = af$ for $f \in L^2(\Omega, \mathcal{F}, \mu)$. We show the following covariant relation.

Proposition 2.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let $\varphi : \Omega \to \Omega$ be a non-singular transformation. If $C_\varphi : L^2(\Omega, \mathcal{F}, \mu) \to L^2(\Omega, \mathcal{F}, \mu)$ is bounded, then we have*

$$C_\varphi^* M_a C_\varphi = M_{\mathcal{L}_\varphi(a)}$$

for $a \in L^\infty(\Omega, \mathcal{F}, \mu)$.

Proof. For $f, g \in L^2(\Omega, \mathcal{F}, \mu)$, we have

$$\begin{aligned}\langle C_\varphi^* M_a C_\varphi f, g\rangle &= \langle M_a C_\varphi f, C_\varphi g\rangle = \int_\Omega a(f \circ \varphi)\overline{(g \circ \varphi)} d\mu \\ &= \int_\Omega a C_\varphi(f\overline{g}) d\mu = \int_\Omega \mathcal{L}_\varphi(a) f\overline{g} d\mu \\ &= \langle M_{\mathcal{L}_\varphi(a)} f, g\rangle\end{aligned}$$

by Lemma 2.1, where C_φ is also regarded as the composition operator on $L^1(\Omega, \mathcal{F}, \mu)$. □

3. C*-algebras associated with complex dynamical systems

We recall the construction of Cuntz-Pimsner algebras [19] (see also [12]). Let A be a C*-algebra and let X be a right Hilbert A-module. A sequence $\{u_i\}_{i=1}^\infty$ of X is called a *countable basis* of X if $\xi = \sum_{i=1}^\infty u_i \langle u_i, \xi\rangle_A$ for $\xi \in X$, where the right hand side converges in norm. We denote by $\mathcal{L}(X)$ the C*-algebra of the adjointable bounded operators on X. For ξ, $\eta \in X$, the operator $\theta_{\xi,\eta}$ is defined by $\theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta\rangle_A$ for $\zeta \in X$. The closure of the linear span of these operators is denoted by $\mathcal{K}(X)$. We say that X is a *Hilbert bimodule* (or *C*-correspondence*) over A if X is a right Hilbert A-module with a $*$-homomorphism $\phi : A \to \mathcal{L}(X)$. We always assume that ϕ is injective.

A *representation* of the Hilbert bimodule X over A on a C*-algebra D is a pair (ρ, V) constituted by a $*$-homomorphism $\rho : A \to D$ and a linear map $V : X \to D$ satisfying

$$\rho(a)V_\xi = V_{\phi(a)\xi}, \quad V_\xi^* V_\eta = \rho(\langle \xi, \eta\rangle_A)$$

for $a \in A$ and $\xi, \eta \in X$. It is known that $V_\xi \rho(b) = V_{\xi b}$ follows automatically (see for example [12]). We define a $*$-homomorphism $\psi_V : \mathcal{K}(X) \to D$ by $\psi_V(\theta_{\xi,\eta}) = V_\xi V_\eta^*$ for $\xi, \eta \in X$ (see for example [10, Lemma 2.2]). A representation (ρ, V) is said to be *covariant* if $\rho(a) = \psi_V(\phi(a))$ for all $a \in J(X) := \phi^{-1}(\mathcal{K}(X))$. Suppose the Hilbert bimodule X has a countable basis $\{u_i\}_{i=1}^\infty$ and (ρ, V) is a representation of X. Then (ρ, V) is covariant if and only if $\| \sum_{i=1}^n \rho(a) V_{u_i} V_{u_i}^* - \rho(a)\| \to 0$ as $n \to \infty$ for $a \in J(X)$, since $\{\sum_{i=1}^n \theta_{u_i,u_i}\}_{n=1}^\infty$ is an approximate unit for $\mathcal{K}(X)$.

Let (i, S) be the representation of X which is universal for all covariant representations. The *Cuntz-Pimsner algebra* $\mathcal{O}_X$ is the C*-algebra generated by $i(a)$ with $a \in A$ and S_ξ with $\xi \in X$. We note that i is known to be injective [19] (see also [12, Proposition 4.11]). We usually identify $i(a)$ with a in A.

Let R be a rational function of degree at least two. We recall the definition of the C*-algebra $\mathcal{O}_R(J_R)$. Since the Julia set J_R is completely invariant under R, that is, $R(J_R) = J_R = R^{-1}(J_R)$, we can consider the restriction $R|_{J_R} : J_R \to J_R$. Let $A = C(J_R)$ and $Y = C(\operatorname{graph} R|_{J_R})$, where $\operatorname{graph} R|_{J_R} = \{(z, w) \in J_R \times J_R \mid w = R(z)\}$ is the graph of $R|_{J_R}$. We denote by $e_R(z)$ the branch index of R at z. Then Y is an A-A bimodule over A by

$$(a \cdot f \cdot b)(z, w) = a(z)f(z, w)b(w), \quad a, b \in A,\ f \in Y.$$

We define an A-valued inner product $\langle\ ,\ \rangle_A$ on Y by

$$\langle f, g\rangle_A(w) = \sum_{z \in R^{-1}(w)} e_R(z)\overline{f(z, w)} g(z, w), \quad f, g \in Y,\ w \in J_R.$$

Then Y is a Hilbert bimodule over A. The C*-algebra $\mathcal{O}_R(J_R)$ is defined as the Cuntz-Pimsner algebra of the Hilbert bimodule $Y = C(\operatorname{graph} R|_{J_R})$ over $A = C(J_R)$.

4. Main theorem

Let R be a rational function. We define the backward orbit $O^-(w)$ of $w \in \hat{\mathbb{C}}$ by

$$O^-(w) = \{z \in \hat{\mathbb{C}} \mid R^{\circ m}(z) = w \text{ for some non-negative integer } m\}.$$

A point w in $\hat{\mathbb{C}}$ is an *exceptional point* for R if the backward orbit $O^-(w)$ of w is finite. We denote by E_R the set of exceptional points.

Definition (Freire-Lopes-Mañé [3], Lyubich [17])**.** Let R be a rational function and $n = \deg R$. Let δ_z be the Dirac measure at $z \in \hat{\mathbb{C}}$. For $w \in \hat{\mathbb{C}} \smallsetminus E_R$ and $m \in \mathbb{N}$, we define a probability measure μ_m^w on $\hat{\mathbb{C}}$ by

$$\mu_m^w = \frac{1}{n^m} \sum_{z \in (R^{\circ m})^{-1}(w)} e_{R^{\circ m}}(z)\delta_z.$$

The sequence $\{\mu_m^w\}_{m=1}^\infty$ converges weakly to a probability measure μ^L, which is called the *Lyubich measure* of R. The measure μ^L is independent of the choice of $w \in \hat{\mathbb{C}} \smallsetminus E_R$.

Let R be a rational function of degree at least two. We will denote by $\mathcal{B}(J_R)$ the Borel σ-algebra on the Julia set J_R. In this section we consider the finite measure space $(J_R, \mathcal{B}(J_R), \mu^L)$. It is known that the support of the Lyubich measure μ^L is the Julia set J_R. Moreover the Lyubich measure μ^L is regular on the Julia set J_R and a invariant measure with respect to R, that is, $\mu^L(E) = \mu^L(R^{-1}(E))$ for $E \in \mathcal{B}(J_R)$. Thus the composition operator C_R on $L^2(J_R, \mathcal{B}(J_R), \mu^L)$ is an isometry.

Definition. For a rational function R of degree at least two, we denote by $\mathcal{MC}_R$ the C*-algebra generated by all multiplication operators by continuous functions in $C(J_R)$ and the composition operator C_R on $L^2(J_R, \mathcal{B}(J_R), \mu^L)$.

Let a rational function R of degree at least two. In this section we shall show that the C*-algebra $\mathcal{MC}_R$ is isomorphic to the C*-algebra $\mathcal{O}_R(J_R)$. First we give a concrete expression of the restriction of $\mathcal{L}_R$ to $C(J_R)$. This result immediately follows from [17] and Lemma 2.1.

Proposition 4.1 (Lyubich [17, Lemma, p.366])**.** *Let R be a rational function of degree n at least two. Then $\mathcal{L}_R : C(J_R) \to C(J_R)$ and*

$$(\mathcal{L}_R(a))(w) = \frac{1}{n} \sum_{z \in R^{-1}(w)} e_R(z)a(z), \quad w \in J_R$$

for $a \in C(J_R)$.

Let $X = C(J_R)$ and $n = \deg R$. Then X is an A-A bimodule over A by

$$(a \cdot \xi \cdot b)(z) = a(z)\xi(z)b(R(z)) \quad a, b \in A,\ \xi \in X.$$

We define an A-valued inner product $\langle\ ,\ \rangle_A$ on X by

$$\langle \xi, \eta \rangle_A(w) = \frac{1}{n} \sum_{z \in R^{-1}(w)} e_R(z)\overline{\xi(z)}\eta(z) \ \left(= (\mathcal{L}_R(\overline{\xi}\eta))(w)\right), \quad \xi, \eta \in X.$$

Then X is a Hilbert bimodule over A. Put $\|\xi\|_2 = \|\langle \xi, \xi \rangle_A\|_\infty^{1/2}$ for $\xi \in X$, where $\| \ \|_\infty$ is the sup norm on J_R. It is easy to see that X is isomorphic to Y as Hilbert bimodules over A. Hence the C*-algebra $\mathcal{O}_R(J_R)$ is isomorphic to the Cuntz-Pimsner algebra $\mathcal{O}_X$ constructed from X.

We need some analyses based on bases of the Hilbert bimodule X to show an equation containing the composition operator C_R and multiplication operators.

Lemma 4.2. *Let $u_1, \dots, u_N \in X$. Then*

$$\sum_{i=1}^N M_{u_i} C_R C_R^* M_{u_i}^* a = \sum_{i=1}^N u_i \cdot \langle u_i, a \rangle_A$$

for $a \in A$.

Proof. Since $a = M_a C_R 1$, we have

$$\begin{aligned}
\sum_{i=1}^N M_{u_i} C_R C_R^* M_{u_i}^* a &= \sum_{i=1}^N M_{u_i} C_R C_R^* M_{u_i}^* M_a C_R 1 \\
&= \sum_{i=1}^N M_{u_i} C_R C_R^* M_{\overline{u}_i a} C_R 1 \\
&= \sum_{i=1}^N M_{u_i} C_R M_{\mathcal{L}_R(\overline{u}_i a)} 1 \qquad \text{by Proposition 2.2} \\
&= \sum_{i=1}^N M_{u_i} M_{\mathcal{L}_R(\overline{u}_i a) \circ R} C_R 1 \\
&= \sum_{i=1}^N u_i \mathcal{L}_R(\overline{u}_i a) \circ R \\
&= \sum_{i=1}^N u_i \cdot \langle u_i, a \rangle_A,
\end{aligned}$$

which completes the proof. □

Lemma 4.3. *Let $\{u_i\}_{i=1}^\infty$ be a countable basis of X. Then*

$$0 \le \sum_{i=1}^N M_{u_i} C_R C_R^* M_{u_i}^* \le I.$$

Proof. Set $T_N := \sum_{i=1}^N M_{u_i} C_R C_R^* M_{u_i}^*$. It is clear that T_N is a positive operator. We shall show $T_N \le I$. By Lemma 4.2,

$$\langle T_N f, f \rangle = \int_{J_R} (T_N f)(z) \overline{f(z)} d\mu^L(z) = \int_{J_R} \left(\sum_{i=1}^N u_i \cdot \langle u_i, f \rangle_A \right)(z) \overline{f(z)} d\mu^L(z)$$

for $f \in C(J_R)$. Since $\{u_i\}_{i=1}^\infty$ is a countable basis of X, for $f \in C(J_R)$, we have $\sum_{i=1}^N u_i \cdot \langle u_i, f \rangle_A \to f$ with respect to $\| \ \|_2$ as $N \to \infty$. Since the two norms $\| \ \|_2$ and $\| \ \|_\infty$ are equivalent (see the proof of [11, Proposition 2.2]), $\sum_{i=1}^N u_i \cdot \langle u_i, f \rangle_A$ converges to f with respect to $\| \ \|_\infty$. Thus

$$\langle T_N f, f \rangle \to \int_{J_R} f(z) \overline{f(z)} d\mu^L(z) = \langle f, f \rangle \quad \text{as } N \to \infty$$

for $f \in C(J_R)$. Therefore $\langle T_N f, f\rangle \leq \langle f, f\rangle$ for $f \in C(J_R)$. Since the Lyubich measure μ^L on the Julia set J_R is regular, $C(J_R)$ is dense in $L^2(J_R, \mathcal{B}(J_R), \mu^L)$. Hence we have $T_N \leq I$. This completes the proof. □

Let $\mathcal{B}(R)$ be the set of branched points of a rational function R. We now recall a description of the ideal $J(X)$ of A. By [11, Proposition 2.5], we can write $J(X) = \{a \in A \,|\, a \text{ vanishes on } \mathcal{B}(R)\}$. We define a subset $J(X)^0$ of $J(X)$ by $J(X)^0 = \{a \in A \,|\, a \text{ vanishes on } \mathcal{B}(R) \text{ and has compact support on } J_R \smallsetminus \mathcal{B}(R)\}$. Since $\mathcal{B}(R)$ is a finite set ([1, Corollary 2.7.2]), $J(X)^0$ is dense in $J(X)$.

Lemma 4.4. *There exists a countable basis $\{u_i\}_{i=1}^\infty$ of X such that*

$$\sum_{i=1}^{\infty} M_a M_{u_i} C_R C_R^* M_{u_i}^* = M_a$$

for $a \in J(X)$.

Proof. By [9, Subsection 3.1], there exists a countable basis $\{u_i\}_{i=1}^\infty$ of X satisfying the following property. For any $b \in J(X)^0$, there exists $M > 0$ such that $\operatorname{supp} b \cap \operatorname{supp} u_m = \emptyset$ for $m \geq M$. Since $J(X)^0$ is dense in $J(X)$, for any $a \in A$ and any $\varepsilon > 0$, there exists $b \in J(X)^0$ such that $\|a - b\| < \varepsilon/2$. Let $m \geq M$. Then by Lemma 4.2 and $bu_i = 0$ for $i \geq m$, it follows that

$$\sum_{i=1}^{m} M_b M_{u_i} C_R C_R^* M_{u_i}^* f = \sum_{i=1}^{m} bu_i \cdot \langle u_i, f\rangle_A = \sum_{i=1}^{\infty} bu_i \cdot \langle u_i, f\rangle_A = bf = M_b f$$

for $f \in C(J_R)$. Since $C(J_R)$ is dense in $L^2(J_R, \mathcal{B}(J_R), \mu^L)$, we have

$$\sum_{i=1}^{m} M_b M_{u_i} C_R C_R^* M_{u_i}^* = M_b.$$

From Lemma 4.3 it follows that

$$\begin{aligned}
\left\| \sum_{i=1}^{m} M_a M_{u_i} C_R C_R^* M_{u_i}^* - M_a \right\| &\leq \left\| \sum_{i=1}^{m} M_a M_{u_i} C_R C_R^* M_{u_i}^* - \sum_{i=1}^{m} M_b M_{u_i} C_R C_R^* M_{u_i}^* \right\| \\
&\quad + \left\| \sum_{i=1}^{m} M_b M_{u_i} C_R C_R^* M_{u_i}^* - M_b \right\| + \|M_b - M_a\| \\
&\leq \|M_a - M_b\| \left\| \sum_{i=1}^{m} M_{u_i} C_R C_R^* M_{u_i}^* \right\| + \|M_a - M_b\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which completes the proof. □

The following theorem is the main result of the paper.

Theorem 4.5. *Let R be a rational function of degree at least two. Then $\mathcal{MC}_R$ is isomorphic to $\mathcal{O}_R(J_R)$.*

Proof. Put $\rho(a) = M_a$ and $V_\xi = M_\xi C_R$ for $a \in A$ and $\xi \in X$. Then we have

$$\rho(a) V_\xi = M_a M_\xi C_R = M_{a\xi} C_R = V_{a\cdot\xi}$$

and

$$V_\xi^* V_\eta = C_R^* M_\xi^* M_\eta C_R = C_R^* M_{\overline{\xi}\eta} C_R = M_{\mathcal{L}_R(\overline{\xi}\eta)} = \rho(\mathcal{L}_R(\overline{\xi}\eta)) = \rho(\langle \xi, \eta\rangle_A)$$

for $a \in A$ and $\xi, \eta \in X$ by Proposition 2.2. Let $\{u_i\}_{i=1}^{\infty}$ be a countable basis of X. Then, applying Lemma 4.4,

$$\sum_{i=1}^{\infty} \rho(a) V_{u_i} V_{u_i}^* = \sum_{i=1}^{\infty} M_a M_{u_i} C_R C_R^* M_{u_i}^* = M_a = \rho(a)$$

for $a \in J(X)$. Since the support of the Lyubich measure μ^L is the Julia set J_R, the $*$-homomorphism ρ is injective. By the universality and the simplicity of $\mathcal{O}_R(J_R)$ ([11, Theorem 3.8]), the C*-algebra $\mathcal{MC}_R$ is isomorphic to $\mathcal{O}_R(J_R)$. $\square$

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